$\mathcal{M}$

full phase space

$\bar{\mathcal{M}}$

Constraint
Surface
$C(x) = C_a(x) = 0$

$\hat{\mathcal{M}}$ = gauge equivalence classes
of points on $\bar{\mathcal{M}}$

reduced phase
space

Definition

$F(q, p)$ is said to be a

strong Dirac observable $\Leftrightarrow$ $\{C(x), F\} = \{C_a(x), F\} = 0$ $\forall x$

weak " " " " on $\bar{\mathcal{M}}$

Quantisation of Systems with first class constraints

- suppose we have first class constraints C_I
- suppose that $\{C_I, C_J\} = f_{IJ}{}^{K} C_K$

structure constants (indep of p, q)

- Choose some sub-Poisson algebra $\mathcal{A}$ of "elementary" variables also closed under complex conjugation

$\mathcal{A}$ should seperate the points on the phase space

- Impose CCR's

$[\hat{a}, \hat{b}] = i\hbar \widehat{\{a, b\}}$

Impose * relations $\hat{a}^{+} = \widehat{\bar{a}}$

$\Rightarrow \hat{\mathcal{A}}$ quantum algebra

Find Hilbert space representation of $\hat{\mathfrak{A}}$

such that $\hat{C}_I$ can be defined as

s.a. operator on $\mathcal{H}$ (Kinematical HS)

and s.t. $[\hat{C}_I, \hat{C}_J] = i\hbar f_{IJ}{}^K \hat{C}_K$

One would like to select physical states

$\hat{C}_I \Psi = 0$, $\Psi \in \mathcal{H}$ (quantum analog of $C_I = 0$)

- Look for solutions as follows

$(C_I \Psi = 0 \quad \|\Psi\|_H = \infty)$

Rigging Map

Find dense, invariant subspace

Φ of $\mathcal{H}$ $\qquad \hat{C}_I \Phi \subset \Phi$

Let Φ^* be the algebraic

dual of Φ : $\Psi \in \Phi^*$ distribution

on Φ, $\Psi[f]$

$\Phi \subset \mathcal{H} \subset \Phi^*$

$\Psi(f) = \langle \Psi, f \rangle$

$\eta : \Phi \to \Phi^*$

case by case Ansatz

$$f \mapsto \int_G d\mu(g) \langle U(g) f, \cdot \rangle$$

$$U(g) = \exp\left(i \sum_I g^I \hat{C}_I\right)$$ unitary operators

μ : Haar measure on the group G generated by the Lie algebra generators C_I

$d\mu(g g_0) = d\mu(g)$

Want to solve instead of $\hat{C}_I \Psi = 0$

$U(g)\Psi = \Psi$

$$\boxed{\Psi(U(g) f) = \Psi(f) \quad \forall f, g}$$

Choose some generators τ_I for Lie Algebra

define $g(t) = \exp(t^I \tau_I)$

$d\mu(g) = d\tilde{\mu}(g(t))$

$$\Psi\left(U(g)f\right)$$

$$= \int d\mu(g') \quad \langle U(g')\Psi, U(g)f \rangle$$

$$= \int d\mu(g') \langle \underbrace{U(g^{-1})U(g')}_{U(g^{-1}g')}\Psi, f \rangle$$

$$= \int d\mu(gg') \langle U(g')\Psi, f \rangle$$

$$= \Psi(f)$$

$$\langle \eta(f), \eta(f') \rangle_{phys}$$

$$= \eta(f)[f'] = \int d\mu(g) \langle U(g) f, f' \rangle_{\mathcal{H}}$$

<u>**Example**</u>

$q_1 \; p_1 \; q_2 \; p_2$; $C = p_2 = 0$; $\mathcal{H} = L_2(\mathbb{R}^2, dx_1 dx_2)$

$$(\hat{p}_2 \psi)(x_1, x_2) = i\hbar \frac{\partial \psi}{\partial x_2}(x_1, x_2) = 0 \Rightarrow \psi = \psi(x_1)$$

However $\|\psi\|^2 = \int dx_1 \, |\psi(x_1)|^2 \underbrace{\left(\int dx_2\right)}_{\infty}$

Let $\phi = \mathcal{S}(\mathbb{R}^2)$ smooth functions of rapid decrease at ∞

$\phi' = \mathcal{S}'(\mathbb{R}^2)$ Schwarz space of tempered distribution

cont. fns on Φ

ϕ^* algebraic dual

$d\mu$

$$\eta(t) = \int \frac{dt}{2\pi} \langle e^{it\hat{p}_2} f, \cdot \rangle$$

$$= \langle \delta(\hat{p}_2) f, \cdot \rangle$$

$$\hat{p}_2 \, \delta(\hat{p}_2) = 0$$

$$\eta(t)\left[e^{it\hat{p}_2} f' \right]$$

$$= \int \frac{dt'}{2\pi} \langle e^{i(t'-t)\hat{p}_2} f, f' \rangle$$

$$= \int \frac{dt}{2\pi} \langle e^{it'} f, f' \rangle$$

$$\langle \eta(f), \eta(f') \rangle_{phys} = \int \frac{dt}{2\pi} \langle e^{it\hat{p}_2} f, f' \rangle_{L_2(d^2x)}$$

$$= \int dt \int d^2x \, \overline{f(x_1, x_2 - t)} \, f'(x_1, x_2)$$

$$= \int dx_1 \, \overline{\left[\int dx_2 \, f(x_1, x_2) \right]} \left[\int dx_2' \, f'(x_1, x_2') \right]$$

$$\Rightarrow \eta(f) \cong \int dx_2 \, f(x_1, x_2)$$

$$\mathcal{H}_{phys} \cong L_2(\mathbb{R}, dx_1)$$