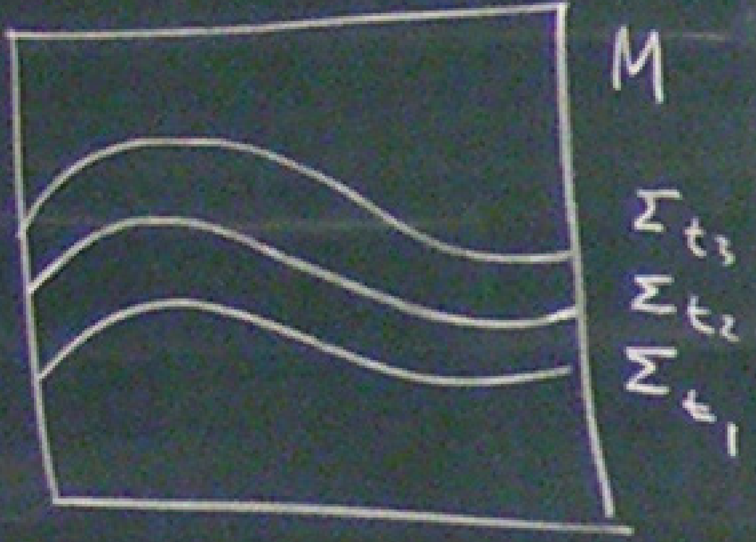
ADM Formulation of GR

$$S = \frac{1}{16\pi G}\int_M d^4x \, \sqrt{|\det g|}\, R^{(4)}$$

$M \cong \mathbb{R} \times \sigma$

Global Hyperbolicity



$X_t : \sigma \longrightarrow \Sigma_t \subset M$

t, x^a ; $a = 1, 2, 3$

$X^\mu(t, x)$ $\quad X_t(x) = X(t, x)$

Lapse shift

$$\frac{\partial X^\mu}{\partial t} = \underset{\text{Lapse}}{N}\, n^\mu + \underset{\text{shift}}{N^a}\, X^\mu_{,a}$$

$$\frac{\partial X^\mu}{\partial x^a}$$

$$g_{\mu\nu}\, n^\mu n^\nu = -1$$

$$g_{\mu\nu}\, n^\mu X^\nu_{,a} = 0$$

city

intrinsic metric

$$\boxed{q_{ab} = g_{\mu\nu}\, X^\mu_{,a} X^\nu_{,b}}$$

$$ds^2 = g_{\mu\nu}\, dX^\mu dX^\nu$$

$$= \left(-N^2 + q_{ab} N^a N^b\right) dt^2 + 2 q_{ab} N^a\, dt\, dx^b + q_{ab}\, dx^a dx^b$$

etric

$$X^\mu_{,a} X^\nu_{,b}$$

$$ab\, dx^a dx^b$$

$$q_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \quad ; \quad q_{\mu\nu} n^\nu = 0$$

$$K_{\mu\nu} = q^\rho_\mu q^\sigma_\nu \nabla_\rho n_\sigma$$

cov. diff
comp. w
$g_{\mu\nu}$

$$K_{\mu\nu} n^\mu = 0$$

$$K_{\mu\nu} = K_{\nu\mu}$$

extrinsic curvature

$$\boxed{K_{ab} = X^\mu_{,a} X^\nu_{,b} K_{\mu\nu} = X^\mu_{,a} X^\nu_{,b} \nabla_\mu n_\nu}$$

$$K_{ab} = \frac{1}{2N}\left(\dot{q}_{ab} - \mathcal{L}_{\vec{N}} q_{ab} \right)$$

ADM Formulation of GR

$$S = \frac{1}{16\pi G}\int_M d^4x \, \sqrt{|\det g|} \; R^{(4)}$$

Gauss + Codazzi

$$K = K_{ab}\, g^{ab}$$

$$R^{(4)}(g) = K_{ab}K^{ab} - K^2 + R^{(3)}(q)$$

+ total divergence

$$S = \underbrace{\frac{1}{16\pi G}}_{\varkappa} \int dt \int d^3x \; N \sqrt{\det(q)} \left(K_{ab} K^{ab} - K^2 + R^{(3)} \right)$$

$$= S[q, \dot{q}, N, \dot{N}, \vec{N}, \dot{\vec{N}}]$$

$$P^{ab} = \frac{\delta S}{\delta \dot{q}_{ab}} = \sqrt{\det q} \left(K^{ab} - K q^{ab} \right)$$

$$P = \frac{\delta S}{\delta \dot{N}} = 0$$

$$P_a = \frac{\delta S}{\delta \dot{N}^a} = 0$$

GR has singular Lagrangean

Cannot solve $N, \dot{N}^a$ for momenta

Canonical Hamiltonian

$$H = \left[\dot{q}_{ab}\, p^{ab} + \dot{N} P + \dot{N}^a P_a - L(q, \dot{q}, N, N^a)\right]$$

$$= \dot{N} P + \dot{N}^a P_a + N^a \mathcal{C}_a + N \mathcal{C}$$

$\dot{q}$'s in terms of p's

$$\mathcal{C}_a = -2 D_b\, p^b{}_a \qquad (D_a \text{ kills } q_{ab})$$

$$\mathcal{C} = \frac{1}{\sqrt{\det q}}\left(p_{ab} p^{ab} - p^2/2\right) - \sqrt{\det q}\, R^{(3)}$$

$$\{ p^{ab}(x), q_{cd}(y) \} = \varkappa \, \delta^a_{(c} \delta^b_{d)} \, \delta(x,y)$$

$$\{ P(x), N(y) \} = \varkappa \, \delta(x,y)$$

$$\{ P_a(x), N^b(y) \} = \varkappa \, \delta(x,y) \, \delta_a{}^b$$

$$\dot{q}_{ab} = \{ \int d^3y \, H(y), q_{ab}(x) \}$$

$$\dot{P}(x) = \{ \int d^3y \, H(y), P(x) \}$$

$$= -\varkappa \, C(x) = 0$$

$$\dot{P}_a(x) = -\varkappa \, C_a(x) = 0$$

secondary constr.

Canonical Hamiltonian is constrained to vanish ∇!

Let $C(f) = \int d^3x \, f(x) \, C(x)$ Ham constr.

$\vec{C}(\vec{f}) = \int d^3x \, f^a(x) \, C_a(x)$ Spatial Diffeo Constr.

<u>Dirac Algebra</u>

$$\{\vec{C}(\vec{f}), \vec{C}(\vec{f}')\} = \chi \, \vec{C}(\mathcal{L}_{\vec{f}} \vec{f}')$$

$$\{\vec{C}(\vec{f}), C(f')\} = \chi \, C(\mathcal{L}_{\vec{f}} f')$$

$$\{C(f), C(f')\} = \chi \int d^3x \, (f f'_{,a} - f' f_{,a}) \, q^{ab} C_b$$

$q_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$

Implementation of Diff(M) in canonical framework

$$\{\vec{C}(\vec{f}),\ q_{\mu\nu}\} = \mathcal{L}_{\vec{f}}\, q_{\mu\nu}$$

$$\{C(f),\ q_{\mu\nu}\} = \mathcal{L}_{fn}\, q_{\mu\nu}$$

N, N^a Lagrange multipliers

have to prescribe them

$$H(N, \vec{N}) = C(N) + C(\vec{N})$$

$$\dot{q}_{ab} = \{H(N,\vec{N}), q_{ab}\} = \ldots$$

$$\dot{p}^{ab} = \{H(N,\vec{N}), p^{ab}\} = \ldots$$

$$C(x) = 0$$

$$C_a(x) = 0$$

Prescribe initial data

q^0_{ab}, p_0^{ab}

Then get

$$G_{\mu\nu} = R^{(4)}_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} = 0$$

with $$ds^2 = (-N^2 + q_{ab} N^a N^b)\, dt^2 + 2 q_{ab} N^b \, dt \, dx^a + q_{ab} \, dx^a dx^b$$